

Five-dimensional metric $f(R)$ gravity and the accelerated universe

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Abstract

The metric $f(R)$ theories of gravity are generalized to five-dimensional spacetimes. By assuming a hypersurface-orthogonal Killing vector field representing the compact fifth dimension, the five-dimensional theories are reduced to their four-dimensional formalism. Then we study the cosmology of a special class of $f(R) = \alpha R^m$ models in a spatially flat FRW spacetime. It is shown that the parameter m can be constrained to a certain range by the current observed deceleration parameter, and its lower bound corresponds to the Kaluza-Klein theory. It turns out that both expansion and contraction of the extra dimension may prescribe the smooth transition from the deceleration era to the acceleration era in the recent past as well as an accelerated scenario for the present universe. Hence five-dimensional $f(R)$ gravity can naturally account for the present accelerated expansion of the universe. Moreover, the models predict a transition from acceleration to deceleration in the future, followed by a cosmic recollapse within finite time. This differs from the prediction of the five-dimensional Brans-Dicke theory but is in consistent with a recent prediction based on loop quantum cosmology.

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1 Introduction

Since 1998, a series of independent observations, including type Ia supernova [1–3], weak lens [4], cosmic microwave background anisotropy [5], large scale structure [6, 7], baryon oscillation [8, 9], etc., have confirmed that our universe is undergoing a period of accelerated expansion. This result conflicts fiercely with the prediction of general relativity, and therefore triggers a “golden age” of cosmology since researchers in cosmology, astrophysics, general relativity, particle physics are all involved in seeking for viable models to explain this phenomena. Within the frame of general relativity, the cosmic speed-up can be viewed as an indication that the present universe is dominated by certain mysterious fluid with large negative pressure, called “dark energy”. However, such simple explanations could hardly be satisfactory. The simplest Λ CDM model, which fits the observation data best so far, assumes a cosmological constant to be responsible for the dark content. But the observed value of the constant Λ is unnaturally smaller than any estimation by tens of orders [10, 11]. Other dynamical dark energy models then waive evolving

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scalar fields to replace and resemble the cosmological constant (for review see [12]). These early attempts provide us with initial understanding of the complexity of the problem, though they chiefly serve for empirical fittings with comparatively poor theoretical motivation, and none of such models turn out to be problem-free.

In view of the challenge and the fact that gravity is the only dominant long-range interaction insofar as we know, it is reasonable to consider the possibility that we have not fully understood the character of gravity on a cosmological scale. Therefore, the exploration of alternative theories of gravity are proposed. One of such examples is the Kaluza-Klein (KK) theory, which was initiated by the motivation of unifying the gravitation field and the electromagnetic field in a five-dimensional (5-D) metric. Since the fifth dimension is supposed to be a compact S^1 circle with an extremely tiny radius, it would actually yield no observable effect. Graceful though, the original version of KK theory fails in passing the Solar-System experiment [13]. But recently a new model of modified (non-compact) Kaluza-Klein cosmology was investigated in [14], where the universe in turns inflates, decelerates, and then accelerates in respectively early times, radiation dominated era and matter dominated era. This result meets the observational facts roughly. But there still exist problems such as that it leaves no decelerated period in the matter-dominant era for the formation of large scale structure. Besides, in [15] Brans-Dicke (BD) theory of gravity was introduced to match the Mach's principle. It is found in [16] that 5-D BD theory can naturally predict the cosmic acceleration without the requirement of a time-varying BD parameter ω or a fabricated potential in its 4-D counterpart [17, 18]. Thus, it lends confidence and motivation to the effort of extending gravitational theories in 4-D to 5-D space-time. Also, note that in [16] the allowed range for parameters $\omega - n$ is considerably wide, where ω is the BD parameter and n represents the interaction between the extra dimension and other spatial ones. We may suspect that further information concerning the different choices of $\omega - n$ are masked in the formalism of 5-D BD theory. Given the argument for the equivalence (by evolutionary effect) of the $f(R)$ theory of gravity to certain special cases of BD theory up to a potential term [19], we are led to consider the 5-D $f(R)$ theories of gravity for such information. It is worth noting that the scale of the compact extra dimension of 5-D gravity theories, which we are considering, is constrained to be less than $50 \mu m$ by current experiments [20–22].

The $f(R)$ theories of gravity extends the Hilbert-Einstein action in general relativity to

$$S = \int d^4x \sqrt{-g} f(R), \quad (1)$$

where $f(R)$ is an arbitrary function of the curvature scalar R . The reason for such exploration in 4-D spacetime is twofold. On the one hand, recent research has shown that a small correction to the Hilbert-Einstein action by adding an inverse term of R would lead to acceleration of the universe [23], and a large variety of models are proposed since then (for review paper, see [24]). However, few of them turn out to be without problem and most of them admit fine-tunings so as to be consistent with experiments. On the other hand, it is highly possible that future quantum theory of gravity, such as string theory, loop quantum gravity, etc., would bring about modifications to the action for classical general relativity, just like that general relativity has introduced corrections to the Newton's gravitation theory. Therefore, this effort would potentially match the low energy effective theory of quantum gravity and therefore accelerate its birth.

With above motivations, we thought that modified gravity without fine-tuning should better be constructed in higher dimensions. It is shown in this paper that values of the parameters in

the 5-D theory are all with reasonable deductions and interpretations. In 5-D spacetimes with a hypersurface-orthogonal Killing vector field, the 5-D $f(R)$ theories of gravity are reduced to the sensible 4-D world in section 2. Here we have adopted KK's idea that the fifth dimension is a small unobservable compact ring S^1 , so that a Killing vector field would arise naturally in the low energy environment. Further, in view of that in KK's theory one can find the sensible 4-D universe to be expanding (without acceleration) as a result of the contraction of the extra dimension, we assume that there is certain interaction between the fifth dimension and other ones at present, and take their relationship to be the general power-law form. Then the reduced theories are carefully studied in the $k = 0$ case of the FRW metric in section 3. We consider a special class of models with $f(R) = \alpha R^m$ and present the numerical simulation of the evolution history of the matter dominated universe in section 4, demonstrating its consistency with the observation. Finally, we end with discussions in section 5 on the similarity and difference between the 5-D $f(R)$ theories and the 5-D BD theory, as well as further indications from our 5-D models.

2 Killing reduction of the 5-D $f(R)$ theory

We start with the action in 5-D space-time:

$$S = \frac{1}{2\kappa} \int d^5x \sqrt{-g} f(R) + S_M. \quad (2)$$

Here $\kappa = 8\pi G^{(5)}/c^4$, where $G^{(5)}$ denotes the gravitational constant in the 5-D spacetime, and S_M represents the matter term in the total action S . Variation of (2) with respect to the metric g^{ab} gives

$$f'(R)R_{ab} - \frac{1}{2}g_{ab}f(R) - (\nabla_a \nabla_b - g_{ab} \nabla^c \nabla_c) f'(R) = \kappa T_{ab}, \quad (3)$$

where $T_{ab} = \frac{-2}{\sqrt{-g}} \frac{\delta S_M}{\delta g^{ab}}$, and $f'(R) = df(R)/dR$. Note that we have employed the abstract index [25] and R, R_{ab}, T_{ab} represent quantities in the 5-D spacetime. Contracting equation (3) with g^{ab} , we obtain the dynamical equation for the scalar field $f'(R)$:

$$\nabla^a \nabla_a f'(R) = \frac{1}{4}[\kappa T - Rf'(R) + \frac{5}{2}f(R)]. \quad (4)$$

Since here $g_{ab}g^{ab}=5$ instead of 4, the dynamical equation of $f'(R)$ differs from its usual 4-D counterpart.

Next we consider the structure of the 5-D spacetime and its relation to our sensible 4-D world. First, as mentioned in the introduction, we assume that the 5-D spacetime possesses a Killing vector field ξ^a which represents the fifth dimension and is everywhere space-like. Recall that the Killing reduction of 4-D spacetime is studied by Geroch [26] and is further extended to the 5-D spacetime by Yang *et al.* [27]. Therefore, following these works we introduce the 5-D metric as

$$g_{ab} = h_{ab} + \lambda^{-1} \xi_a \xi_b, \quad (5)$$

where h_{ab} is the metric in the usual 4-D universe and $\lambda = \xi^a \xi_a$. If ξ^a is not hypersurface-orthogonal, it would be related to electromagnetic 4-potential in the reduced 4-D theory, as demonstrated by KK theory. Since we are chiefly concerned with the cosmological effect of the reduced model, our discussion is restricted to the case where ξ^a is hypersurface-orthogonal for

convenience. Without losing generality, a coordinate system can be chosen as $\{x^\mu, x^5\}$, $\mu = 0, 1, 2, 3$, with $(\frac{\partial}{\partial x^5})^a = \xi^a$. Then the line element of g_{ab} reads $ds^2 = g_{\mu\nu}dx^\mu dx^\nu + \lambda dx^5 dx^5$, while Ricci tensors and the dynamical equation of λ in 5-D and 4-D spacetime bear the following relation [27]:

$$R_{ab}^{(4)} = \frac{1}{2}\lambda^{-1}D_a D_b \lambda - \frac{1}{4}\lambda^{-2}(D_a \lambda)D_b \lambda + h_a^m h_b^n R_{mn}, \quad (6)$$

$$D^a D_a \lambda = \frac{1}{2}\lambda^{-1}(D^a \lambda)D_a \lambda - 2R_{ab}\xi^a \xi^b. \quad (7)$$

Here D_a denotes the covariant derivative on 4-D spacetime and is defined as [25]

$$D_a T_{b_1 \dots b_n}^{c_1 \dots c_m} = h_a^d h_{b_1}^{e_1} \dots h_{b_n}^{e_n} h_{f_1}^{c_1} \dots h_{f_m}^{c_m} \nabla_d T_{e_1 \dots e_n}^{f_1 \dots f_m},$$

satisfying all the conditions for a derivative operator. On the other hand, the stress-energy tensor in (3) is regarded as a perfect fluid in 5-D spacetime with the expression [16]

$$T_{ab}^{(5)} = L^{-1}\lambda^{-\frac{1}{2}}[(\rho + P)U_a U_b + P g_{ab}] = L^{-1}\lambda^{-\frac{1}{2}}[T_{ab}^{(4)} + P\lambda\xi_a \xi_b], \quad (8)$$

where ρ and P are the 4-D energy density and the hydrostatic pressure respectively, and L is a constant representing the coordinate scale of the fifth dimension. This extension of the stress-energy tensor of the 4-D perfect fluid to a 5-D one can be understood in the following way: because the fifth dimension is compact and attached to every point of the 4-D space-time, ρ and P experienced by a 4-D observer should be an integrated effect throughout the compact ring. If we further expect that the fluid distributes homogeneously and does not travel along the fifth dimension, it is clear that $\rho = \int \rho^{(5)} \lambda^{1/2} dx^5 = \rho^{(5)} \lambda^{1/2} L$, $P = P^{(5)} \lambda^{1/2} L$, and hence equation (8) is obtained. Combining (3), (4) with (6), (7), and (8), straightforward calculations lead to the 4-D field equation:

$$\begin{aligned} G_{ab}^{(4)} &= \frac{8\pi G}{c^4} \frac{\lambda^{-\frac{1}{2}}}{f'(R)} T_{ab}^{(4)} + h_{ab} \frac{f(R) - Rf'(R)}{2f'(R)} \\ &+ \frac{1}{2}\lambda^{-1}(D_a D_b - h_{ab} D^c D_c)\lambda - \frac{1}{4}\lambda^{-2}[(D_a \lambda)D_b \lambda - h_{ab}(D^c \lambda)D_c \lambda] \\ &+ \frac{1}{f'(R)}(D_a D_b - h_{ab} D^c D_c)f'(R) - \frac{1}{2f'(R)}\lambda^{-1}h_{ab}(D^c \lambda)D_c f'(R), \end{aligned} \quad (9)$$

and the dynamical equations of λ and $f'(R)$:

$$\begin{aligned} D^a D_a \lambda &= \frac{8\pi G}{c^4} \frac{\lambda^{\frac{1}{2}}}{f'(R)} \left(\frac{1}{2}T^{(4)} - \frac{3}{2}p \right) + \frac{1}{2}\lambda^{-1}(D^a \lambda)D_a \lambda - \frac{1}{f'(R)}(D^a \lambda)D_a f'(R) \\ &+ \frac{\lambda}{f'(R)} \left[\frac{1}{4}f(R) - \frac{1}{2}Rf'(R) \right], \end{aligned} \quad (10)$$

$$D^a D_a f'(R) = \frac{8\pi G}{c^4} \frac{\lambda^{-\frac{1}{2}}}{4}(T^{(4)} + p) - \frac{1}{2}\lambda^{-1}(D^a \lambda)D_a f'(R) + \frac{1}{4} \left[\frac{5}{2}f(R) - Rf'(R) \right], \quad (11)$$

where $G = G^{(5)}L^{-1}$ represents the usual 4-D gravitational constant [28].

3 FRW cosmology of the reduced $f(R)$ gravity

In this section, we will study the cosmological predictions of the reduced $f(R)$ gravity. The metric of a spatially isotropic and homogeneous 4-D spacetime is the Friedman-Robertson-Walker (FRW) metric with three possible structures of the space. As suggested by the observation [29], we only handle the spatially flat case where the 4-D line element reads

$$ds^2 = -dt^2 + a^2(t) \sum_{i=1}^3 dx_i^2. \quad (12)$$

Then the two components of the field equation (9) are:

$$3 \left(\frac{\dot{a}}{a} \right)^2 = 8\pi G \frac{\lambda^{-\frac{1}{2}}}{f'(R)} \rho - \frac{3}{2} \frac{\dot{a}}{a} \frac{\dot{\lambda}}{\lambda} - 3 \frac{\dot{a}}{a} \frac{f''(R) \dot{R}}{f'(R)} - \frac{1}{2} \frac{\dot{\lambda}}{\lambda} \frac{f''(R) \dot{R}}{f'(R)} - \frac{f(R) - Rf'(R)}{2f'(R)} \quad (13)$$

and

$$\begin{aligned} - \left(2 \frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} \right) &= \frac{8\pi G}{c^2} \frac{\lambda^{-\frac{1}{2}}}{f'(R)} P + \frac{\dot{a}}{a} \frac{\dot{\lambda}}{\lambda} - \frac{1}{4} \left(\frac{\dot{\lambda}}{\lambda} \right)^2 + \frac{1}{2} \frac{\dot{\lambda}}{\lambda} \frac{f''(R) \dot{R}}{f'(R)} + 2 \frac{\dot{a}}{a} \frac{f''(R) \dot{R}}{f'(R)} \\ &\quad + \frac{\ddot{\lambda}}{\lambda} + \frac{f'''(R) \dot{R}^2 + f''(R) \ddot{R}}{f'(R)} + \frac{f(R) - Rf'(R)}{2f'(R)}. \end{aligned} \quad (14)$$

The dynamical equations of λ and $f'(R)$ are respectively:

$$\frac{\ddot{\lambda}}{\lambda} = 8\pi G \frac{\lambda^{-\frac{1}{2}}}{f'(R)} \left(\frac{\rho}{2} \right) - 3 \frac{\dot{a}}{a} \frac{\dot{\lambda}}{\lambda} + \frac{1}{2} \left(\frac{\dot{\lambda}}{\lambda} \right)^2 - \frac{\dot{\lambda}}{\lambda} \frac{f''(R) \dot{R}}{f'(R)} - \frac{f - 2Rf'(R)}{4f'(R)}, \quad (15)$$

and

$$\frac{f'''(R) \dot{R}^2 + f''(R) \ddot{R}}{f'(R)} = 8\pi G \frac{\lambda^{-\frac{1}{2}}}{f'(R)} \left(\frac{\rho}{4} - \frac{P}{c^2} \right) - 3 \frac{\dot{a}}{a} \frac{f''(R) \dot{R}}{f'(R)} - \frac{1}{2} \frac{\dot{\lambda}}{\lambda} \frac{f''(R) \dot{R}}{f'(R)} - \frac{\frac{5}{2}f(R) - Rf'(R)}{4f'(R)}. \quad (16)$$

Here an over-dot denotes the derivative with respect to the proper time t of the isotropic observer, and we have assumed that the scalar field λ depends only on t . Note that the conservation equation of the 5-D stress-energy tensor, $\nabla^\mu T_{\mu 0} = 0$, gives:

$$\dot{\rho} + 3 \frac{\dot{a}}{a} \left(\rho + \frac{P}{c^2} \right) + \frac{1}{2} \frac{\dot{\lambda}}{\lambda} \frac{P}{c^2} = 0, \quad (17)$$

which can also be obtained directly from equations (13) – (16). For the common baryonic matter content of the present universe, the pressure is negligible compared to the energy density. Thus, we set $P/c^2 = 0$ by approximation in the above equation and get $\rho = \rho_0(a_0/a)^3$, where ρ_0 is the current observed energy density of the luminary matter.

By now, we have been dealing with the general form of $f(R)$ gravity. Next we restrict our discussion to a specific class of $f(R)$ models, $f(R) = \alpha R^m$, to further study the evolutionary characteristics of the theory. It has been shown that this choice in 4-D spacetime, with $m \neq 1$

and the common baryonic matter, could fit the Hubble diagram of Type Ia supernovae without need of dark energy [30]. We denote $f'(R) = \alpha m R^{m-1} \equiv \phi$ and obtain

$$R = \left(\frac{\phi}{\alpha m} \right)^{\frac{1}{m-1}}, \quad f(R) = \frac{\phi}{m} \left(\frac{\phi}{\alpha m} \right)^{\frac{1}{m-1}}, \quad (18)$$

where α is a dimensional constant. With the above analysis, we find the three cosmological evolution equations for numerical simulations from the combination of equations (13) – (16):

$$\frac{\ddot{a}}{a} = \left(\frac{\dot{a}}{a} \right)^2 + \frac{\dot{a}}{a} \frac{\dot{\lambda}}{\lambda} + 2 \frac{\dot{a}}{a} \frac{\dot{\phi}}{\phi} + \frac{1}{2} \frac{\dot{\lambda}}{\lambda} \frac{\dot{\phi}}{\phi} + \left(\frac{3}{8m} - \frac{1}{4} \right) \left(\frac{\phi}{\alpha m} \right)^{\frac{1}{m-1}} - 8\pi G \lambda^{-\frac{1}{2}} \phi^{-1} \frac{3}{4} \rho_0 \left(\frac{a_0}{a} \right)^3, \quad (19)$$

$$\frac{\ddot{\lambda}}{\lambda} = -3 \frac{\dot{a}}{a} \frac{\dot{\lambda}}{\lambda} + \frac{1}{2} \left(\frac{\dot{\lambda}}{\lambda} \right)^2 - \frac{\dot{\lambda}}{\lambda} \frac{\dot{\phi}}{\phi} - \left(\frac{1}{4m} - \frac{1}{2} \right) \left(\frac{\phi}{\alpha m} \right)^{\frac{1}{m-1}} + 8\pi G \lambda^{-\frac{1}{2}} \phi^{-1} \frac{1}{2} \rho_0 \left(\frac{a_0}{a} \right)^3, \quad (20)$$

$$\frac{\ddot{\phi}}{\phi} = -3 \frac{\dot{a}}{a} \frac{\dot{\phi}}{\phi} - \frac{1}{2} \frac{\dot{\lambda}}{\lambda} \frac{\dot{\phi}}{\phi} - \left(\frac{5}{8m} - \frac{1}{4} \right) \left(\frac{\phi}{\alpha m} \right)^{\frac{1}{m-1}} + 8\pi G \lambda^{-\frac{1}{2}} \phi^{-1} \frac{1}{4} \rho_0 \left(\frac{a_0}{a} \right)^3. \quad (21)$$

4 Numerical simulations

Let us now find the natural initial values of $a_0, \dot{a}_0, \lambda_0, \dot{\lambda}_0, \phi_0, \dot{\phi}_0$ for the numerical simulation. Firstly, a_0, λ_0 themselves have no direct physical meaning and therefore can be simply fixed as 1 (with no dimension). Then we can directly determine the value of \dot{a}_0 by the present value of Hubble parameter $H_0 = (\dot{a}/a)_{t_0}$. Secondly, we adopt the dynamical compactification idea of KK cosmology that the contraction of the extra dimension would result in the expansion of the remaining dimensions [31], and assume

$$a^3 \lambda^{n/2} = \text{constant}, \quad (22)$$

where for a particular $f(R)$ theory with given m and α in Eq.(18), the free parameter n can be constrained by the observation. Therefore we have $(\dot{\lambda}/\lambda)_{t_0} = 6H_0/n$. Finally, comparing the matter term in equation (9) to the field equation of general relativity, we expect $\lambda^{-1/2} \phi^{-1} \sim 1$, at least for the present period; thus $\phi_0 = 1$ (with no dimension either) and further $\left(\frac{\dot{\phi}}{\phi} \right)_{t_0} = -\frac{1}{2} \left(\frac{\dot{\lambda}}{\lambda} \right)_{t_0}$. In summary, the initial conditions are:

$$\begin{cases} a_0 = \lambda_0 = \phi_0 = 1, \\ \dot{a}_0 = H_0, \dot{\lambda}_0 = -\frac{6}{n} H_0, \dot{\phi}_0 = \frac{3}{n} H_0. \end{cases} \quad (23)$$

Then we estimate the value of α in equations (19) – (21) and determine the allowed range for the pair of parameters m and n . A calculation of the 5-D curvature scalar shows

$$R = 6 \left(\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} \right) + \frac{\ddot{\lambda}}{\lambda} + \frac{1}{2} \left(\frac{\dot{\lambda}}{\lambda} \right)^2 + 3 \frac{\dot{a}}{a} \frac{\dot{\lambda}}{\lambda}. \quad (24)$$

Inserting (18), (20) into the expression of R and noting that the deceleration parameter $q = -\frac{\ddot{a}a}{\dot{a}^2}$ is an observable with its present value denoted as q_0 , we have

$$\left(\frac{1}{\alpha m} \right)^{\frac{1}{m-1}} = \frac{24m}{1+2m} \left[1 - q_0 + \frac{9}{n^2} + \frac{8\pi G \rho_0}{12H_0^2} \right] H_0^2. \quad (25)$$

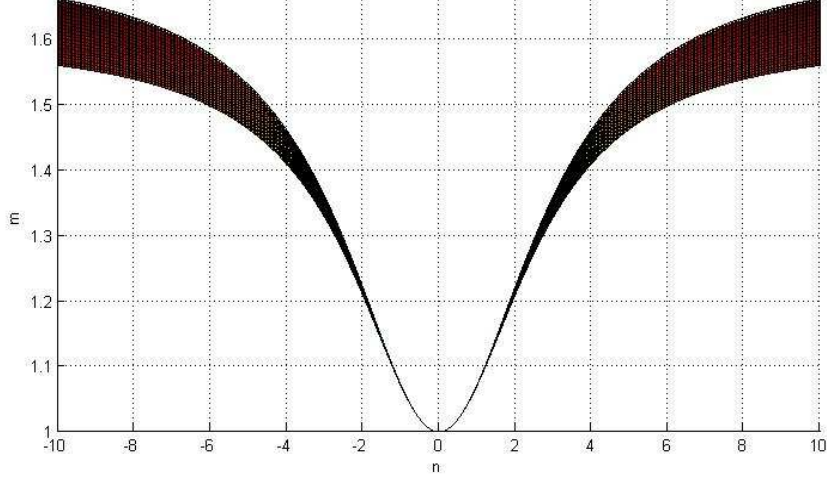


Figure 1: Allowed range for the values of m and n with $q_0 \in (-0.47, -0.67)$ [32]. The horizontal axis stands for different value of n , while the vertical axis represents m .

Substituting (25) into (19), we obtain the relationship between parameters m and n as follows:

$$m = \frac{5 - 4q_0 + \frac{36}{n^2}}{2(1 - 2q_0) + \frac{36}{n^2} + \frac{8\pi G\rho_0}{H_0^2}}, \quad (26)$$

where $G = 6.67 \times 10^{-11} \text{ kg}^{-1} \text{ m}^3 \text{ s}^{-2}$, $\rho_0 = (3.8 \pm 0.2) \times 10^{-28} \text{ kg m}^{-3}$, $H_0 = (2.3 \pm 0.1) \times 10^{-18} \text{ s}^{-1}$ [32] and therefore $\frac{8\pi G\rho_0}{H_0^2} \approx 0.12$. One necessary and sufficient condition for the present accelerated expansion of the universe is that the deceleration parameter $q_0 < 0$. Specifically, the allowed range for q_0 according to the present observation is $q_0 = (-0.57 \pm 0.10)$ [32]. Here we apply this criteria to determine the allowed range for m and n , as depicted in Fig. 1. From the figure, it is obvious that there do exist suitable parameters m and n , that is, suitable formalisms of $f(R) = \alpha R^m$ and corresponding relationships between the extra dimension and the other ones, such that the present cosmic acceleration can be explained without the need of dark energy. Besides, Fig. 1 also reveals that both positive and negative values of n is allowed, indicating that the idea of dynamical compactification can be extended to that both the expansion and the contraction of the extra dimension can be responsible for the accelerated expansion of the other dimensions. Besides, we know from equation (25) that

$$\alpha = \frac{1}{m} \left[\frac{24m}{1+2m} \left(1 - q_0 + \frac{9}{n^2} + \frac{8\pi G\rho_0}{12H_0^2} \right) H_0^2 \right]^{-(m-1)}. \quad (27)$$

Using Eqs. (26) and (27), we depict the $\alpha - n$ relation in Fig. 2 with the range spanned by q_0 , from which we find α increases rapidly as $|n|$ increases. Further, one could find from Fig. 1 or equation (26) that the lower bound of m is 1, corresponding to $\alpha = 1$, and therefore $f(R) = R$, which is exactly the KK case. One subtlety concerning this case is that Fig. 1 as well as equation (26) shows that $n = 0$ for $m = 1$, and therefore equation(22) apparently indicates $H_0 = \dot{a} = 0$, contradicting with the present observations. Thus one might think that the KK theory is inconsistent with the observation. However, this conclusion is misleading, because

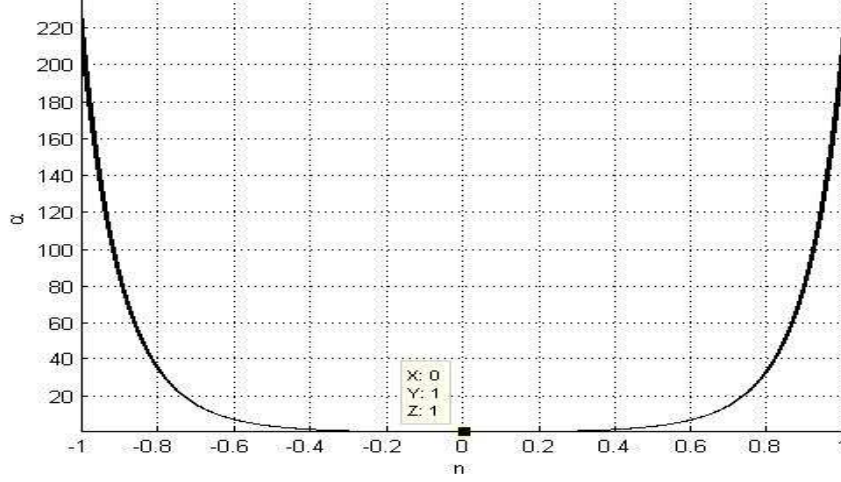


Figure 2: The relation between n and the possible range for α . The range is spanned by the uncertainty of the deceleration parameter q_0 .

from the inverse-solving procedure in equation (18), we have already confined our discussion to $m \neq 1$ cases. On the other hand, the asymptotic value of m , as n approaches infinity, is

$$\lim_{n \rightarrow \infty} m = \frac{5 - 4q_0}{2(1 - 2q_0) + 0.12}, \quad (28)$$

from which we find the upper bound for m to be 1.72. Hence, the possible range for m is narrowed down as $1 < m < 1.72$.

With equations (19) – (21), initial values (23) and (25), the numerical simulation of the evolution of the scale factor $a(t)$, the scalar field $\lambda(t), \phi(t)$ and the deceleration parameter $q(t)$ can be conducted. As an example, we choose $m = 1.5, n = 5.4$. Then from equation (26) we get $q_0 = -0.6$. The corresponding value of $\left(\frac{1}{\alpha m}\right)^{\frac{1}{m-1}}$ is approximately $17.3H_0^2$. It is useful to define the effective gravitational “constant” from Eqs. (9) – (11) as

$$\tilde{G} = G\lambda^{-1/2}f'(R)^{-1} = G\lambda^{-1/2}\phi^{-1}. \quad (29)$$

Note that \tilde{G} is actually an evolving scalar. Then the evolution characters of $a(t), q(t), \lambda(t), \phi(t)$ and \tilde{G} are illustrated in Figs. 3 – 6 respectively. From these figures, it is clear that the present deceleration parameter $q_0 = -0.6$ is in consistent with observation and the above analysis. Moreover, we note the following:

1. $q(t)$ rolls from a positive value to a negative one smoothly in the recent past. Specifically, if the age of the universe is 13.5Gyr [32], the universe turns from deceleration to acceleration at $t = (13.5 - 1.2)\text{Gyr}$. Thus, the “coincidence” problem in dark energy can be addressed in this case.
2. After $q(t)$ reaches the minimal value, it rolls back again and becomes positive at $t = (13.5 + 13.4)\text{Gyr}$. Thus, the universe will become decelerating in the future rather than be endlessly accelerating.

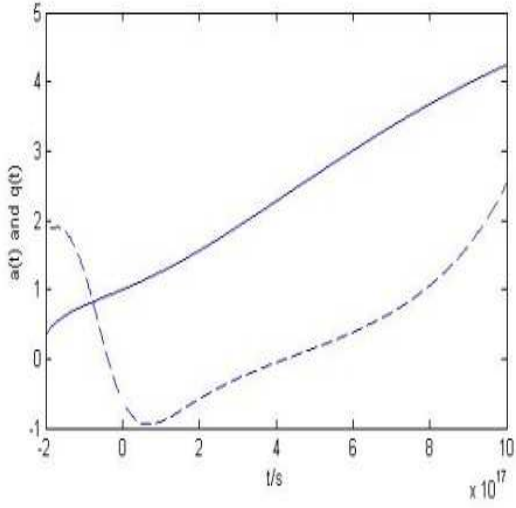


Figure 3: The evolution of $a(t)$ (solid line) and $q(t)$ (dashed line) with $m = 1.5, n = 5.4$. Here $t=0$ corresponds to today, and the present value of the deceleration parameter is $q_0 = -0.6$.

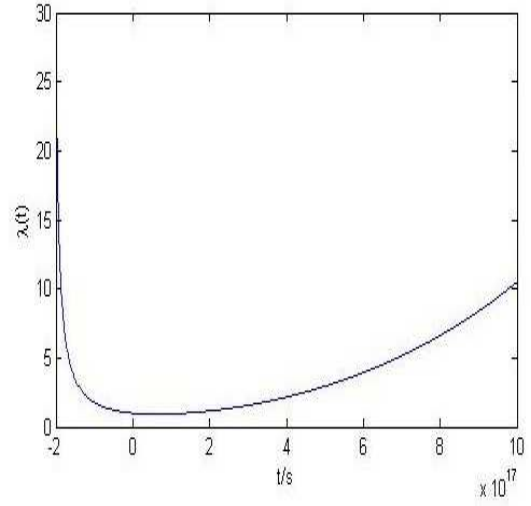


Figure 4: The evolution of $\lambda(t)$ with $m = 1.5, n = 5.4$. Note that $\lambda(t)$ is decreasing at $t = 0$.

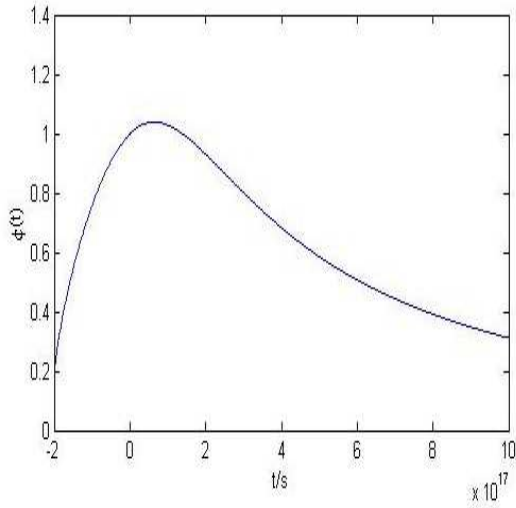


Figure 5: The evolution of $\phi(t)$ with $m = 1.5, n = 5.4$.

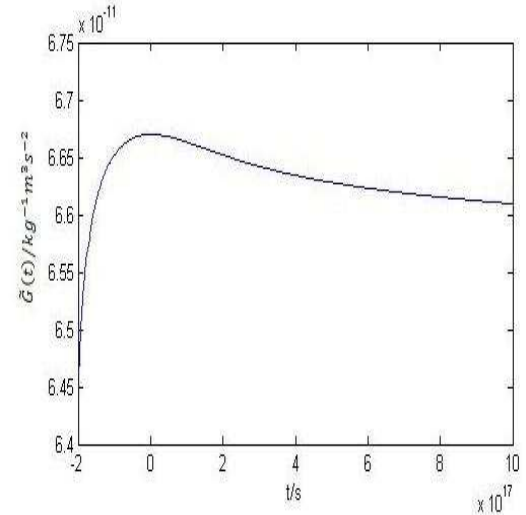


Figure 6: The evolution of $\tilde{G}(t)$ with $m = 1.5, n = 5.4$.

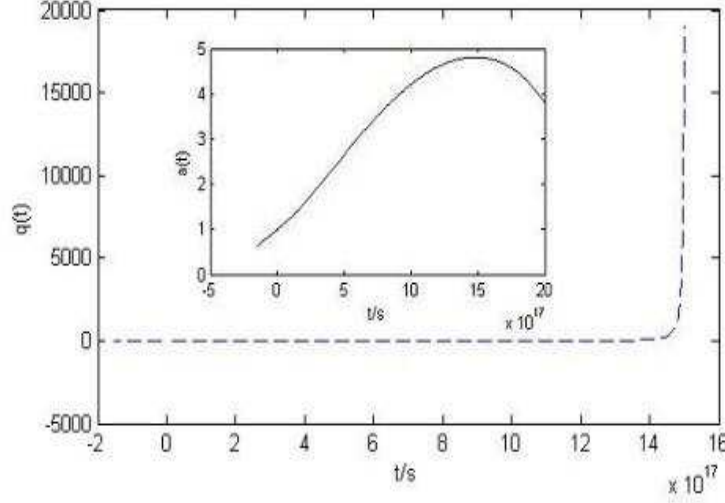


Figure 7: Numerical simulation for a longer period of cosmic evolution. The deceleration parameter $q(t)$ becomes divergent at $t = (13.5 + 47.6)\text{Gyr}$ (dashed line), as $a(t)$ (solid line in the subfigure) approaches a constant value.

3. Numerical simulation for a long period of cosmic evolution shows that the value of $q(t)$ becomes divergent at $t = (13.5 + 47.6)\text{Gyr}$, as depicted in Fig. 7. Since $a(t) > 0$ and its evolution is generally slow, the only reason for the divergence of $q(t)$ is $\dot{a} \rightarrow 0$. In other words, the universe would come to a static point in the finite future. Actually in this period, numerical simulation shows that all quantities, such as $\lambda(t)$ and $\phi(t)$, still remain well-behaved. After this static point, $a(t)$ starts to decrease for a certain period before numerical simulation fails. Thus, around the static point of $a(t)$, numerical simulation is still effective.
4. $\lambda(t)$ decreases from a large value in the past and increases slowly at present, while $\phi(t)$ increases to a maximum value for the present period and decreases slowly in the future. After the static point of $a(t)$, nonetheless, the decrease of $\phi(t)$ becomes faster and therefore it would approach 0 in the finite future when the numerical simulation fails. Actually, since ϕ appears in the denominators of Eqs. (19) – (21), the analysis from this set of equations is no longer applicable for the evolution character beyond the $\phi = 0$ point. Instead, one should take advantage of original equation (3) (or Eqs. (9) – (11) by multiplying $f'(R)$ on both sides). We leave the details about the static point and the evolution beyond to future investigations.
5. The evolution of \tilde{G} is generally slow in the future. The present value of \tilde{G} is maximal with an almost zero evolving speed. Thus, it is also in consistent with the observation data that the present gravitational constant has a negligible evolving speed.

On the other hand, as indicated by Fig. 1, the expansion of the extra dimension would also bring about the acceleration of other dimensions. If we take $m = 1.5, n = -5.4$ instead of $n = +5.4$, the numerical simulation will give similar evolution features of $a(t), q(t)$ and $\tilde{G}(t)$ with the present value of deceleration parameter $q_0 = -0.6$, which are consistent with observations. In

this case, the universe turned from deceleration to acceleration at $t = (13.5 - 6.1)\text{Gyr}$, which was 4.9Gyr earlier than that in the $n = 5.4$ case. Correspondingly, the future transition point of the universe from acceleration to deceleration is also moved ahead to $t = (13.5 + 8.0)\text{Gyr}$, rendering that the total acceleration period of the universe changes only moderately from 14.6Gyr to 14.1Gyr. Thus, the $n < 0$ case is very similar to the $n > 0$ case, except that the cosmic transition from deceleration to acceleration and the inverse occur both ahead of time. The reason for such temporal translation can be viewed in the following way. As shown before, when $1 < m < 1.72$, the future asymptotic evolution of λ should be increasing. Therefore, if the initial (present) evolution rate for λ is set to be increasing at present rather than be decreasing, there should be a temporal translation with respect to the evolution of λ and therefore to other quantities that can be affected.

5 Discussions

To clarify the argument for the equivalence of the $f(R)$ theory of gravity to certain special cases of BD theory up to a potential term [19], we now compare the cosmic evolution of 5-D $f(R)$ models developed in previous sections to the 5-D BD theory [16]. From equation (9), it is noticeable that the potential term is $\frac{f(R) - Rf'(R)}{f'(R)}$, and similar terms also exist in equations (10) and (11). These terms correspond to the $(\frac{\phi}{\alpha m})^{\frac{1}{m-1}}$ terms in (19) – (21), and would be shown to play significant roles in predicting the future evolution of the universe. There are three independent functions in the 5-D BD theory [16], namely $a(t)$, $\lambda(t)$ and $\phi(t)$, where $\phi(t)$ represents the core idea of BD theory on an evolving gravitation “constant” due to the interaction between the local gravitational fields with the faraway matter in the universe. We note that $q(t)$ in this case (see Fig. 2 in [16]) evolves to a constant in the future. This means that the universe would accelerate permanently in the future until its energy density decreases to be extremely low so that certain quantum effects may become significant, such as the situation in [33]. It can also be read from Figs. 3 – 5 in [16] that the asymptotic values of $\frac{\dot{\lambda}}{\lambda}$ and $\frac{\dot{\phi}}{\phi}$ are both zero, as further illustrated in Fig. 8.

In contrast, in our 5-D $f(R)$ theories, Fig. 5 has demonstrated that, while ϕ evolves to a large value presently and remains positive in the future, the $(\frac{\phi}{\alpha m})^{\frac{1}{m-1}}$ terms render the evolution of $\lambda(t)$, $\phi(t)$ to be more complicated. Specifically, the corresponding evolution characteristics of $\frac{\dot{a}}{a}$, $\frac{\dot{\lambda}}{\lambda}$, $\frac{\dot{\phi}}{\phi}$ are illustrated in Fig. 9 with specific parameters $m = 1.5, n = 5.4$. As $\tilde{G} = G\lambda^{1/2}\phi^{-1}$ is an observable, it is interesting to show how $(\frac{\phi}{\alpha m})^{\frac{1}{m-1}}$ terms affect its evolution. Since $m > 1$, $q_0 < 0$, from equation (25) we have $(\frac{1}{\alpha m})^{\frac{1}{m-1}} > 0$. Moreover, we do not consider any solution with a negative ϕ because of its anti-gravity character. Thus, the coefficient of the $(\frac{\phi}{\alpha m})^{\frac{1}{m-1}}$ term would determine the evolutionary inclination of λ and ϕ . Specifically, because $m = 0.5$ and $m = 2.5$ correspond to $(\frac{1}{4m} - \frac{1}{2}) = 0$ and $(\frac{5}{8m} - \frac{1}{4}) = 0$ respectively, the qualitative features of the future evolution of λ and ϕ are as follows:

- If $m < 0.5$, $\lambda \downarrow$, $\phi \downarrow$
- If $m > 2.5$, $\lambda \uparrow$, $\phi \uparrow$
- If $0.5 < m < 2.5$, $\lambda \uparrow$, $\phi \downarrow$

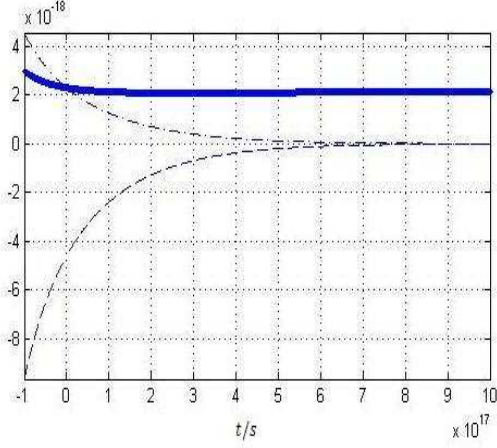


Figure 8: The evolution rate of $\frac{\dot{a}}{a}$ (solid line), $\frac{\dot{\lambda}}{\lambda}$ (dashed line) and $\frac{\dot{\phi}}{\phi}$ (dash-dotted line) in the 5-D BD theory, with parameters $\omega = 1.2, n = 3$ in [16].

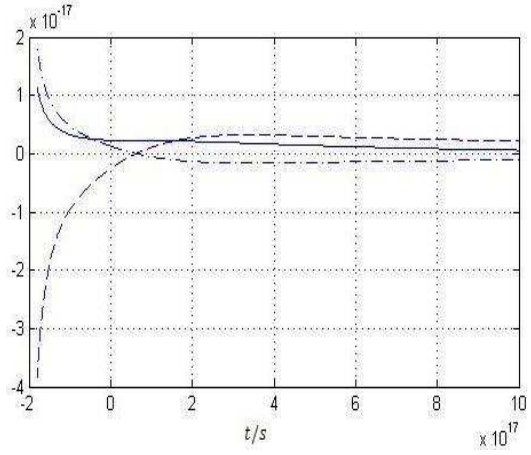


Figure 9: The evolution of $\frac{\dot{a}}{a}$ (solid line), $\frac{\dot{\lambda}}{\lambda}$ (dashed line) and $\frac{\dot{\phi}}{\phi}$ (dash-dotted line) in 5-D $f(R)$ gravity with parameters $m = 1.5, n = 5.4$.

In view of the constraint illustrated by Fig. 1, only the last situation is allowed, where the evolution of λ and ϕ counterbalances each other and therefore the evolution of \tilde{G} is generally slow.

In summary, we present the Killing reduction of 5-D $f(R)$ theories of gravity to the 4-D sensible world. Then we study its cosmological implication by assuming the spatial homogeneity and isotropy, namely, the Friedman-Robertson-Walker metric. With the illustration of a specific example it is found that the theory is consistent with the present observations in a variety of aspects, including the recent cross from cosmic deceleration to acceleration, its prescription of present cosmic speed-up, the negligible evolving speed of the effective gravitational constant, and etc.. In contrast to the KK cosmology, it is worth mentioning that in 5-D $f(R)$ theories of gravity, both expansion and contraction of the extra dimension could result in the present accelerated expansion of other spatial dimensions. Even for a simple and generic class of $f(R) = \alpha R^m$ models, the 5-D $f(R)$ theories of gravity do not need unreasonable or ill-initiated fine-tuning of certain parameters. Hence it is reasonable to infer that the present accelerated expansion of spatial dimensions could be certain basic character of the 5-D spacetimes.

Finally, it is pointed out in Ref. [33] that in the large scale limit, quantum gravity effect would come to be vital and bring about higher-order quantum corrections to the Friedman equation to prescribe a static point in the finite future, followed by the recollapse of the universe. Note that the static point in the finite future is also a significant character of the 5-D $f(R)$ gravity. Thus, it is of interest to explore whether there are some relation between the 5-D theories of $f(R) = \alpha R^m$ models and the quantum gravity theory applied in [33]. In addition, the future recollapse also differs from the prediction of endless acceleration scenario in the 5-D BD cosmology [16]. Hence, 5-D $f(R)$ gravity is not simply equivalent to 5-D BD theory, since the effect of potential terms are shown to be vital enough to prescribe different cosmological scenarios in the distant future.

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